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Number theory

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Problem Set #6

Recall that: a module P is projective if and only if for every surjective module homomorphism $f : M \to P$ there exists a module homomorphism $g : P \to M$ such that $f \circ g = id_P$.

Exercise 10 p 23 of [N]
The fractional ideal $\mathfrak{a}$ of a Dedekind domain $\mathcal{O}$ are projective $\mathcal{O}$-modules, i.e given any surjective homomorphism $f : M \to N$ of $\mathcal{O}$-modules, each homomorphism $g : \mathfrak{a} \to N$ can be lifted to a homomorphism $h : \mathfrak{a} \to M$ such $f \circ h = g$.
Solution:
Given a surjective homomorphism $f : M \to \mathfrak{a}$ of $\mathcal{O}$-modules. We need to construct a homorphism $g : \mathfrak{a} \to M$ such $f \circ h = Id_M$. We know that in a Dedekind domain any fractional ideal is invertible. In particular, $\mathfrak{a}$ is invertible so that there is $\mathfrak{a}^{-1}$ such that $\mathfrak{a}^{-1}\mathfrak{a} = \mathcal{O}$. In particular, there is $x_i \in \mathfrak{a}$ and $y_i \in \mathfrak{a}^{-1}$, $1 = \sum_{i=1}^n x_i y_i$. As f is surjective, we can pick $e_i \in M$ with $f(e_i) = x_i$. Now define the following map:

$$\begin{array}{rccl} g: & \mathfrak{a} & \to & M \\ & x & \mapsto & \sum_{i=1}^n (xy_i)e_i \end{array}$$

We then calculate for $x \in \mathfrak{a}$:

$$\begin{array}{rcl} f \circ g(x) & = & f(\sum_{i=1}^n (xy_i)e_i) \\ & = & \sum_{i=1}^n (xy_i)f(e_i) \\ & = & x\sum_{i=1} (x_i y_i) \\ & = & x \end{array}$$

Exercise 6 p 38 of [N]
Let $\mathfrak{a}$ be an integral ideal of K and $\mathfrak{a}^m = (a)$. Show that $\mathfrak{a}$ becomes a principal ideal in the field $L = K(\sqrt[m]{a})$, in the sense that $\mathfrak{a}\mathcal{O}_L = (\alpha)$.
Solution:
If $m = 1$ the result is obvious. Otherwise, $(\mathfrak{a}\mathcal{O}_L)^m = \mathfrak{a}^m\mathcal{O}_L = (a)\mathcal{O}_L = (a)$. Now, clearly $(\sqrt[m]{a})^m = (a)$. So that $(\mathfrak{a}\mathcal{O}_L)^m = (a^{1/m})^m$ as ideal of L. Finally, $(\sqrt[m]{a}) = \mathfrak{a}\mathcal{O}_L$, from the unicity of the decomposition in prime ideals.

Exercise 7 p 38 of [N]:
Show that, for every number field K, there exists a finite extension L such that every ideal of K becomes a principal ideal.
Solution:

Since Cl_K is finite, let $m = |Cl_K|$, we can consider a finite number of representative of the class in Cl_K. Let say I_1, ..., I_m and by Lagrange theorem, $[I_i^m] = e$ that is there is $a_i \in K$ such that $I_i^m = (a_i)$, for any $1 \leq im$ and by the previous exercise, I_i become all principal in the finite extension $L = K(\sqrt[m]{a_1}, ..., \sqrt[m]{a_m})$. Now, let I a ideal of K, then $I \sim I_i$, for some i, so that there is a $a \in K$, such that $I = (a)I_i$, and $I = (a\sqrt[m]{a_i})$ in L. As a consequence, any